MATH 2028 Honours Advanced Calculus II
2021-22 Term 1
Problem Set 7
due on Nov 8, 2021 (Monday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Gradescope on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Problems to hand in

1. Compute the line integral $\int_C F \cdot d\vec{r}$ where
   (a) $F(x,y) = (-x^2y, xy^2)$ and C is the circle of radius $a > 0$ centered at the origin, oriented counterclosewise;
   (b) $F(x,y) = (-y^2, x^2)$ and C is the boundary of the region given in polar coordinates by $r \leq a$, $0 \leq \theta \leq \pi/4$ oriented counterclosewise.
2. Let C be the circle $x^2 + y^2 = 2x$ oriented counterclosewise. Evaluate the line integral $\int_C F \cdot d\vec{r}$ where
$$F(x,y) = \left(-y^2 + e^{x^2}, x + \sin(y^3)\right).$$
3. Let $0 < b < a$. Find the area under the curve $f(t) = (at - b\sin t, a - b\cos t)$, $0 \leq t \leq 2\pi$, above the x-axis.

Suggested Exercises

1. Compute the line integral $\int_C F \cdot d\vec{r}$ where
   (a) $F(x,y) = (-y\sqrt{x^2+y^2}, x\sqrt{x^2+y^2})$ and C is the circle $x^2 + y^2 = 2x$ oriented counterclosewise;
   (b) $F(x,y) = (-y^3, x^3)$ where C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ oriented counterclosewise.
2. Find the area of the region enclosed by the curve $x^{2/3} + y^{2/3} = 1$.
3. Find the area of the region enclosed by the curve
$$\gamma(t) = \big(\cos t + t\sin t, \sin t - t\cos t\big), \quad 0 \leq t \leq 2\pi$$
   and the line segment from $(1, -2\pi)$ to $(1, 0)$.
4. Suppose C is a piecewise C^1 closed curve in $\mathbb{R}^2$ that intersects with itself finitely many times and does not pass through the origin. Show that the line integral
$$\frac{1}{2\pi}\int_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$
   is always an integer. This is called the *winding number* of C around the origin.

Challenging Exercises

1. Give a direct proof of Green's theorem for

   (a) a triangle with vertices $(0,0)$, $(a,0)$ and $(0,b)$,

   (b) the region $\{(x,y) : a \leq x \leq b, g(x) \leq y \leq h(x)\}$ for some C^1 function $g, h : [a,b] \to \mathbb{R}$.